Reward and Risk in the Fixed Income Markets

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Abstract
In this work we deal the portfolio optimization problem in the fixed income market considering two different risk sources: price risk and market risk. In this framework we propose a three decisional subsequent steps model. In particular, we take into account the price risk discussing an implementation of the classical immunization problem applied to the U.S. fixed income market. The portfolio selection model implements both classic and innovative immunization methodologies. Moreover, using the ex-post results from the optimal immunization problem we are able to account the market risk with other optimization procedures. The results show that we can build efficient portfolios that minimize the price risk and the market risk.

Keywords: Fixed-Income market, portfolio selection, immunization, risk analysis
JEL codes: G11,G12

1. Introduction

Bond portfolio management evolved a lot over the last decades. The objectives of a bond portfolio manager are mainly three: immunization, active management and passive management. With active management strategies the investor tries to achieve a capital gain trading fixed income assets. Trading strategies are defined using a term structure model which summarizes the expectations about the market. Passive management is generally applied with buy-and-hold strategies and/or benchmark replication strategies. With immunization strategies the manager control the price risk and try to increase the portfolio return. Moreover, almost all these strategies serves to increase the return and to limit the price risk of a portfolio of bonds.

In this paper we discuss how to separate and deal the different risk sources in portfolio problems. In particular, we present a general theory and an unifying framework for understanding how to control and optimize the risk in portfolio problems applied to the fixed income market. Moreover, we propose an empirical application of portfolio selection models which account different risk sources.

There exist several sources of risk in the markets: market risk, credit risk, inflation risk, liquidity risk, price risk, currency risk, reinvestment risk, etc... In this paper we consider the price risk and the market risk which have more impact than other risk sources and for which we discuss an empirical application on the US fixed income market.

As for the stocks, one of the most important risk for the bonds is the variability of the prices. In the case of bonds the price variability is mainly associated to three factors: the variations of the interest rate term structure, the credit risk of the issuer and the so-called systematic risk (also called market risk). Fixed income investors model the price risk due to the variations of the interest rates using the basic principles of immunization (Fisher and Weil, 1971, Macaulay and Durand, 1951, Redington , 1952, Weil, 1973, De Felice, 1995, Munk, 2011). We observe that, once we get the ex-post observations of optimal immunized portfolios with fixed duration, we can use these portfolios as funds with fixed duration and, then, we can propose classic portfolio models that account the market risk among the different funds. Investors optimize a reward-risk performance measure (Rachev et al., 2008, Cogneau and Hübner, 2009) to control the market risk. Therefore, optimizing a reward-risk performance measure among the optimal funds of bonds we are able to propose a two step portfolio selection model that accounts both the risk sources (the market risk and the price risk).

Section 2 discusses a general portfolio selection methodology which takes into account different risk sources. Section 3 examines in practice the proposed portfolio selection models applied to the US fixed income market. In Section 4 we summarize the results.

2. Portfolio selection considering different risk sources

In this section we deal different risk sources in portfolio problems on the fixed income market. In particular, we analyze two risk sources: price risk and market risk. We discuss the basic principles of immunization theory (De Felice, 1995, Macaulay and Durand, 1951) used to control the price risk of a given portfolio of bonds and we present some simple methods to control the market risk valued on historical data. Moreover, we introduce a new formulation of a financial immunization method based on the new concept of immunization in average.

The portfolio selection problem is traditionally studied in terms of return and risk performance. However, the concept of return and risk for treasury and corporate bonds is different from other securities. The classic return and risk measures used for corporate and sovereign are respectively the portfolio future wealth (given by the product between the initial wealth and the yield to maturity) and the modified duration of the portfolio.

2.1 Portfolio problem considering the price risk in the fixed income market

In order to describe the classic methods used to control the price risk we should recall the basic principles of immunization theory introduced by Fisher and Weil (1971) and Redington (1952). The financial immunization is defined as a method that reduces the risk produced by the variation in the interest rates. A classical way to achieve this is to make the asset and the liability portfolios as similar as possible and so equally sensitive to interest rate changes.

Considering a fixed rate of return r, the price today of a bond P that will imply n future payments c_k at time $t_k, k = 1, \dots, n$, is simply given by the formula:

$$P = \sum_{k=1}^{n} \frac{c_k}{(1+r)^{t_k}} \quad (1)$$

Generally the market required rate of return is not fixed over time and the rate r for which equality (1) holds is called yield to maturity of the bond P.

To introduce some immunization theorems we show how a change in the interest rates impacts in the price using the Taylor's first and second order approximation. Taylor's first order approximation involves the modified duration:

$$\varDelta P \approx -D\varDelta rP \tag{2}$$

where D is the modified duration which is defined as:

$$D = -\frac{1}{P}\frac{\partial P}{\partial r} = \frac{\sum_{k=1}^{n}(t_k \cdot c_k(1+r)^{-t_k})}{P(1+r)} \tag{3}$$

Observe that in literature the modified duration is often called duration while we call Macaulay duration the formula given by $\widehat{D} = D(1+r)$ (Macaulay and Durand, 1951 and Weil, 1973). The Taylor's second order approximation is:

$$\varDelta P \approx \left(-D\varDelta r + \frac{1}{2}C(\varDelta r)^2\right)P \tag{4}$$

where C is the convexity defined as:

$$C = \frac{1}{P}\frac{\partial^2 P}{\partial r^2} = \frac{\sum_{k=1}^{n}\left(t_k \cdot c_k(1+r)^{-t_k}\right) + \sum_{k=1}^{n}\left(t_k^2 \cdot c_k(1+r)^{-t_k}\right)}{P \cdot (1+r)^2} \tag{5}$$

There exist several possible approximation formulas of the convexity. In this paper we use the approximation suggested by DataStream based on the Macaulay duration and the yield to maturity i.e.

$$\frac{\partial^2 P}{\partial r^2} \approx \left[\left(\frac{100+Y}{100+Y_1}\right)^{\widehat{D}} + \left(\frac{100+Y}{100+Y_2}\right)^{\widehat{D}} - 2\right] \cdot 10^8 \tag{6}$$

where Y is the yield to maturity, Y_1 the yield plus one basis point (0.01%) and Y_2 the yield minus one basis point.

Fisher and Weil (1971) based their immunization theorem on the first order approximation. Redington (1952) introduced the immunization for infinitesimal shift in the interest rate by matching the assets and liabilities duration and having the assets convexity greater than the liabilities convexity. Vasicek and Fong (1984) generalized immunization for each type of additive shift in the yield curve.

In portfolio problems we can account immunization theory to control the price risk due to the variation of the interest rates. Thus, assume that $r = [r_1, \dots, r_n]'$ is the vector of the yields to maturity of the assets, $D = [D_1, \dots, D_n]'$ is the vector of the modified durations of each nominal bond, $C = [C_1, \dots, C_n]'$ is the vector of the convexity, and $\nu = [\nu_1, \dots, \nu_n]'$ is the vector of the wealth invested in the bonds i.e. $\nu_i = y_i P_i$ where y_i is the number of assets invested in the *i-th* bond and P_i is the price of the *i-th* bond. Then, according to Fisher and Weil's approach, we consider, as return measure, the expected future wealth approximated by $\nu'(1+r)$ and, as risk measure, the modified duration of the portfolio approximated by the formula $D_{(p)} = \frac{\nu' D}{W}$ where $W = \sum_{i=1}^{n} \nu_i$ is the initial wealth. Thus, when no short sales are allowed ($y_i \geq 0$) and supposing that cannot be invested more than 21% in a unique asset $\left(\frac{y_i P_i}{W} \leq 0.21\right)$, the investor that wants to minimize the risk of variation of the prices, will choose a solution of the following optimization problem:

$$max_y \, v'(1+r)$$

$$subject\ to \quad \begin{cases} \sum_{i=1}^{n} y_i P_i = W \\ \frac{v'D}{W} = d \\ y_i P_i = v_i \\ y_i \geq 0 \\ \frac{y_i P_i}{W} \leq 0.21 \end{cases} \tag{7}$$

For $i = 1, \dots, n$ and for some fixed modified duration d and an initial wealth W. According to the Redington's approach (1952) we have to add in portfolio problem (7) the constraints that the convexity of the portfolio at time t is higher than the convexity of the portfolio at time $t-1$, that is :

$$C_t(ptf) = \frac{v_t' C_t}{W_t} \geq C_{t-1}(ptf) = \frac{v_{t-1}' C_{t-1}}{W_{t-1}} \tag{8}$$

Observe that in portfolio problems of type (7) we do not need historical observations of bond returns to estimate the return and risk measure.

2.2 The Portfolio problem in the fixed income market considering the immunization in average

In this section we introduce the definition of immunization in average to model a portfolio selection problem where the average of the asset returns grows after changes of the yield term structure. Let us consider that the term structure change of a random additive shift Δr. We want to understand if there exist alternative definitions of risk that account the average of the term structure changes and it guarantees that the future wealth will increase when the term structure change. With this aim we first give the following definition.

Definition 1 *Given two portfolios one of activities and the other of liabilities that at a given time have the same value, we say that the portfolio of assets is immunized in average if its average is greater than the average value of the liability portfolio after changes of the term structure.*

Requiring the *immunization in average* of an asset portfolio can be guaranteed or by using the classic Redington theorem or by considering the following lemma.

Lemma 1 *Given two portfolios A of assets and L of liabilities that at a given time have the same value. The portfolio of assets is immunized in average by an infinitesimal random additive shift Δr if the liability portfolio has the same risk measure given by*

$$D - \frac{1}{2}E(\Delta r)C \tag{9}$$

and the convexity of the asset portfolio is greater than the convexity of the liability portfolio.

Proof: Let us consider the Taylor approximation

$$\begin{aligned} &\frac{\Delta P}{P} \approx -D\Delta r + \frac{1}{2}C(\Delta r)^2 - \frac{1}{2}E(\Delta r)C\Delta r + \frac{1}{2}E(\Delta r)C\Delta r \\ &\Rightarrow \frac{\Delta P}{P} \approx -\left(D - \frac{1}{2}E(\Delta r)C\right)\Delta r + \frac{1}{2}C(\Delta r - E(\Delta r))\Delta r \end{aligned} \tag{10}$$

Computing the average we obtain the following formulation:

$$E\left(\frac{\Delta P}{P}\right) = -\left(D - \frac{1}{2}E(\Delta r)C\right)E(\Delta r) + \frac{1}{2}C\sigma_{\Delta r}^2 \tag{11}$$

Thus, given risk measure $\varphi = D - \frac{1}{2}E(\Delta r)C$, if we compute the average of the difference between the future returns of assets (*A*) and liabilities (L), we get the thesis:

$$E\left(\frac{\Delta P^A}{P} - \frac{\Delta P^L}{P}\right) = -(\varphi^A - \varphi^L)E(\Delta r) + \frac{1}{2}(C^A - C^L)\sigma_{\Delta r}^2 \geq 0 \qquad (12)$$

Observe that generally the immunization in average does not guarantees the classic immunization concept. As a matter of fact, under the hypothesis that $\varphi^A = \varphi^L$ and $C^A > C^L$, then by (10) holds:

$$\left(\frac{\Delta P^A}{P} - \frac{\Delta P^L}{P}\right) = +\frac{1}{2}(C^A - C^L)(\Delta r - E(\Delta r))\Delta r: \qquad (13)$$

That is lower than zero any time either $0 < \Delta r < E(\Delta r)$ or $E(\Delta r) < \Delta r < 0$. However, the lower guaranties of immunization in average permits to account in an unique risk measure φ the convexity and the average of the term structure changes. This aspect a priori could add flexibility to the portfolio choice that could increment the investor's future wealth.

Therefore, in Section 3 we suggest to manage the price risk taking into account the immunization in average. In particular, we maximize the future wealth fixing the new risk measure $\varphi = D - \frac{1}{2}E(\Delta r)C$ and requiring that the portfolio convexity at a given time is greater or equal to the one of the previous time, as in the following optimization problem:

$$\begin{aligned} & max_y \, v'(1+r) \\ & subject\ to \begin{cases} \sum_{i=1}^n y_i P_i = W \\ \frac{v'\varphi}{W} = d \\ y_i P_i = v_i \\ y_i \geq 0 \\ \frac{y_i P_i}{W} \leq 0.21 \\ C_t(ptf) = \frac{v_t' C_t}{W_t} \geq C_{t-1}(ptf) = \frac{v_{t-1}' C_{t-1}}{W_{t-1}} \end{cases} \end{aligned} \qquad (14)$$

Once we solve problems of type (7) or (14) and we compute the ex post wealth for different fixed modified durations, we can use the ex-post observations as historical observations of funds with fixed duration shifting in the time the optimal portfolio selection problem. In particular, we can apply this procedure to reduce the market risk computing other optimal portfolios funds with fixed duration.

2.3 Portfolio problem considering the market risk

To deal the market risk is generally used a reward-risk portfolio selection model (Rachev et al., 2008) applied either to historical series or to simulated scenarios. Thus, let us consider n assets with returns $r = [r_1, \dots, r_n]'$ and a benchmark with return r_b. We denote by $x = [x_1, \dots, x_n]'$ the vector of the positions taken in the n assets. Then the return portfolio is given by $x'r = \sum_{i=1}^n x_i r_i$. In order to maximize the performance of a portfolio in the reward-risk framework we generally provide the maximum expected reward ν per unit of risk ρ. This optimal portfolio is commonly referred to as the market portfolio and is obtained by maximizing the ratio between the reward and the risk when both are positive measures. There exist several possible performance ratios $G(X) = \frac{\nu(X)}{\rho(X)}$ which can be used in portfolio choices. The most important characteristic is the isotony with an order of preference; that is, if X is preferable to Y then $G(X) \geq G(Y)$. Although the financial literature on investor behavior agrees that investors are non-satiable, there is not a common vision about the investors' aversion to risk. Thus investors' choices should be isotonic with non-satiable investors' preferences (i.e., if $X \geq Y$ then $G(X) \geq G(Y)$). Several behavioral finance studies suggest that most investors are neither risk averse nor risk loving. A first classification with respect to the different characteristics of return and risk measures is given in Rachev et al. (2008), Cogneau, and Hübner (2009a and 2009b).

Here we review two performance measures we will use in the next portfolio analysis: the Sharpe Ratio and the Rachev Ratio. According to Markowitz' mean-variance analysis, Sharpe (1994) suggested that investors should maximize what is now referred to as Sharpe Ratio (SR) given by

$$SR(x'r) = \frac{E(x'r - r_b)}{STD(x'r - r_b)} \tag{15}$$

where r_b is a benchmark return and $STD(x'r - r_b)$ is the standard deviation of excess returns. Maximizing the Sharpe Ratio, we get a market portfolio that should be optimal for non-satiable risk-averse investors, and that is not dominated in the sense of second-order stochastic dominance. This performance measure is fully compatible with elliptically distributed returns, but it will lead to incorrect investment decisions when returns present heavy tails or skewness. In order to account of heavy tails and skewness several other performance ratios based on tail measures have been proposed. In contrast to the Sharpe Ratio, the Rachev Ratio is based on tail measures and it is isotonic with the preferences of non-satiable investors that are neither risk averse nor risk lovers.

The Rachev Ratio (RR) is the ratio between the average of earnings and the mean of losses; that is,

$$RR(x'r, \alpha, \beta) = \frac{CVaR_\beta(r_b - x'r)}{CVaR_\alpha(x'r - r_b)} \tag{16}$$

where the Conditional Value-at-Risk (CvaR), is a coherent risk measures (Artzner et al., 1999) defined as

$$CVaR_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha VaR_q(X)dq \tag{17}$$

and

$$VaR_q(X) = -F_X^{-1}(q) = -inf\{x|P(X \leq x) > q\} \tag{18}$$

is the Value-at-Risk (VaR) of the random return X. If we assume a continuous distribution for the probability law of X, then $CVaR_\alpha(X) = -E\big(X\big|X \leq VaR_\alpha(X)\big)$ and, therefore CVaR, can be interpreted as the average loss beyond VaR. Typically we use historical observations of returns to estimate a return and a risk measure of the return portfolio. A consistent estimator of $CVaR_\alpha(X)$ is given by:

$$CVaR_\alpha(X) = \frac{-1}{[\alpha M]}\sum_{i=1}^{[\alpha M]} X_{i:n} \tag{19}$$

where M is the number of historical observations of X, $[\alpha M]$ is the integer part of αM and $X_{i:M}$ is the *i-th* observation of X ordered in increasing values. Similarly an approximation of $VaR_q(X)$ is simply given by $-X_{[qM]:M}$. Once we are able to approximate the portfolio return and risk measures, we can apply portfolio selection optimization problems to the approximated portfolio of returns.

Therefore, when no short sales are allowed ($x_i \geq 0$) and supposing that cannot be invested more than 21% in a unique asset ($x_i \leq 0.21$), we assume that the investors will choose the market portfolio solution to the following optimization problem

$$\begin{aligned} &max_x\, G(x'r) \\ &subject\ to \quad \begin{cases} \sum_{i=1}^{n} x_i = 1 \\ x_i \geq 0 \\ x_i \leq 0.21 \end{cases} \end{aligned} \tag{20}$$

where $G(x'r)$ is either the Sharpe Ratio or the Rachev Ratio.

3. An ex-post empirical analysis

In this section we propose an ex-post empirical application. In particular, we first examine our dataset taken from DataStream. Secondly, we propose an ex-post empirical analysis for few classes of credit risk and we try to guarantee a minimum of liquidity of the bond portfolios. In particular, at each recalibration time we consider only those bonds that were traded during the last week. In order to guarantee this minimum level of liquidity we require that the average value of the traded volume overcome at least the 10,000 USD. Then for preselected classes of credit risk and fixed liquidity constraints we apply the two different portfolio selection problems to reduce the price risk and the market risk. Finally we analyze and discuss the results of the optimal portfolio procedures.

3.1 The dataset

In order to consider a more realistic analysis of the US fixed income market we consider a dynamic dataset that accounts all the treasury and corporate bonds present in DataStream during the decade 2002-2012. In the ex-post portfolio analysis we recalibrate weekly (every 5 trading days) the portfolio. The bonds used at each recalibration time could be different since we use a dynamic dataset. The dataset is dynamic because it considers expired, living and issued bonds during the period from January 2002 till September 2012. In total for all period we have a dataset of 43,835 bonds. Moreover, for each bond we have daily information about price, volume, Moody's classification rating, yield to maturity (also called internal rate of return – IRR), Macaulay duration and modified duration. The yield to maturity and the Macaulay duration are used to approximate the convexity of each bond (as in formula (5)). From this first set of bonds we exclude those which presents either a too high variability or unjustified values of duration, yield to maturity and prices. In particular, we consider the following filters:

a) We remove bonds near to default or with non-justifiable prices. We discard also securities whose distance between the lower price and top price were too high: the maximum price must be less than five times the minimum price (price variability constraint).

b) We remove bonds with negative or constant Macaulay duration or modified duration. Then for expired bonds we remove those for which in the observed period the gap between the maximum and the minimum duration or modified duration is greater than fifteen years. For living securities we remove those with duration or modified duration grater then fifty (duration constraints).

c) About IRR as lower bound we reject bonds with IRR less than minus two and at the same time durations greater than four months. (IRR constraints)

We remark that the major part of the cleaning is due to the negativity of the duration and the negativity of the IRR.

In this way we remove 3,662 bonds and obtain a dataset composed by 40,173 bonds. Finally, we remove all bonds without rating (the majority) and we take only bonds with rating at least B3: we allow the presence of high volatility bonds but we don't want to deal with bonds with rating of type C (from Caa1 to C) which are too risky and difficult to manage. Finally we compose the dataset of bonds with rating of type A or B (from Aaa to B3), in total 10.000 securities and we call it dataset AB.

3.2 The Portfolio selection problem with price risk and market risk

In order to consider the market risk and the price risk due to the variations of the interest rates, we distinguish two phases for the portfolio decisional process where the two different risk sources are controlled.

In the first phase we weekly compute the ex-post wealth we obtain optimizing the expected future wealth of the portfolio of bonds taking into account the price risk. We use the Fisher and Weil approach (1971) for determining optimal portfolios with fixed modified durations and then we create a fund of bonds for any fixed modified duration among 60 possible values.

In the second phase we use as historical series the ex-post wealth of the 60 funds we obtain in the first phase. Then we optimize two performance measures (Sharpe ratio and Rachev ratio) on these series. With this second phase we practically reduce the market risk obtaining portfolios of the funds obtained in the first phase. Again the composition of the portfolio changes weekly.

3.2.1 Portfolio selection that account the price risk

According to Section 2 we can control the price risk due to the interest rate variations optimizing the portfolio yield to maturity for some fixed duration. In this portfolio selection problem we recalibrate the portfolio composition in a dynamic dataset of bonds. Clearly, at each recalibration time, the dataset is composed just by the living securities in that time.

In particular, we consider 60 fixed risk values in the interval [5-18] year. The risk values are either the modified durations or $\varphi = D - \frac{1}{2}E(\Delta r)C$. We do not use risk measures less than 5 because a sufficient number of assets with risk values less than 5 do not always exist for all the ex-post period. The funds change in their composition periodically to maintain constraint the portfolio duration. In particular, the recalibration occurs weekly (every 5 trading days) starting the 1st January 2002 until 13 September 2012. We assume an initial wealth equal to 100,000 USD, no short sales are allowed and we cannot invest more than 21% in a single asset.

Practically, for each rating class we have to compute the optimal portfolio composition 558 times and at the *k-th* optimization ($\mathrm{k} = 1, 2, \dots, 558$) three main steps are performed to compute the ex-post final wealth:

Step 1 – Preselect all the liquid and active assets in the last six months (125 trading days).

Step 2 – Determine the optimal portfolio y that maximizes the final wealth for a fixed duration, i.e. the solution of the optimization problems (7) or (14) for $i = 1, \dots, n$. Moreover, this is a linear optimization problem that can be computed in a very efficient way.

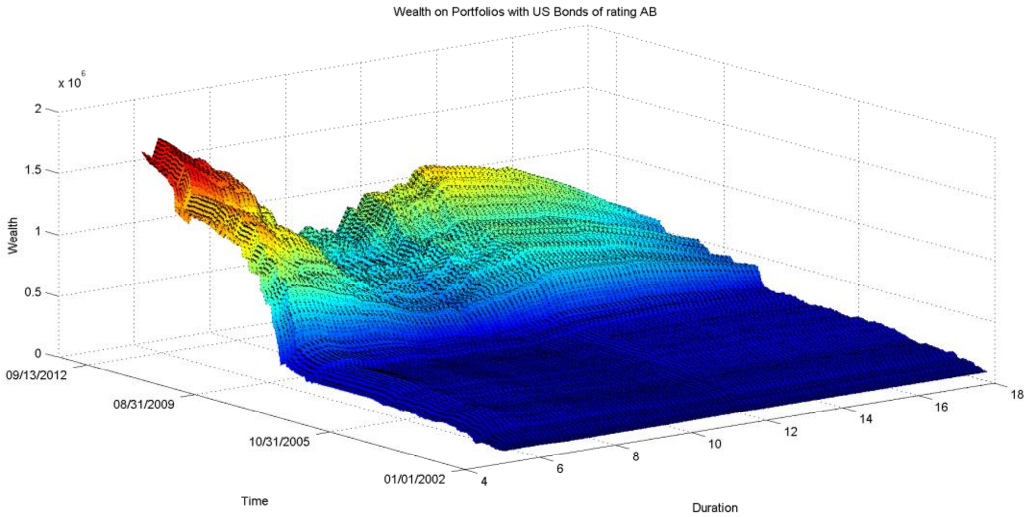
Step 3 – Compute the ex-post final wealth taking into account of 5 basis points proportional transaction costs.

Steps 1, 2, 3 are repeated for all the 60 durations until some observations are available.

The results of this ex-post comparison are reported in Figures 1 and Figure 2. Generally, for all different risk values we notice that the wealth remains low until the crisis and start to increase mainly after 2009. On the one hand, during the crisis period all the bonds suffer negative shocks. This means that we have a huge market risk and we need to reduce this risk sources. On the other hand, it appears that in 2009 born a bubble in the US fixed income market that is not over yet at the end of 2012. The only asset bubble-free seems to be the very high rating bonds with short modified duration.

Moreover, in Figure 1 we observe the results of the classical portfolio optimization problem using the modified duration as risk measure. The figure illustrates that we have the best results in terms of wealth with the shortest modified durations and we can achieve a wealth of more than 1,500,000 USD. This result seems counterintuitive if we think that generally to higher risk positions correspond higher returns. However the dataset contains all the bonds of different ratings and those with lowest rating generally give higher returns. Since low rating classes present a very high risk of default then the investor choose to invest in these classes only for lower levels of modified duration in order to reduce the risk of default.

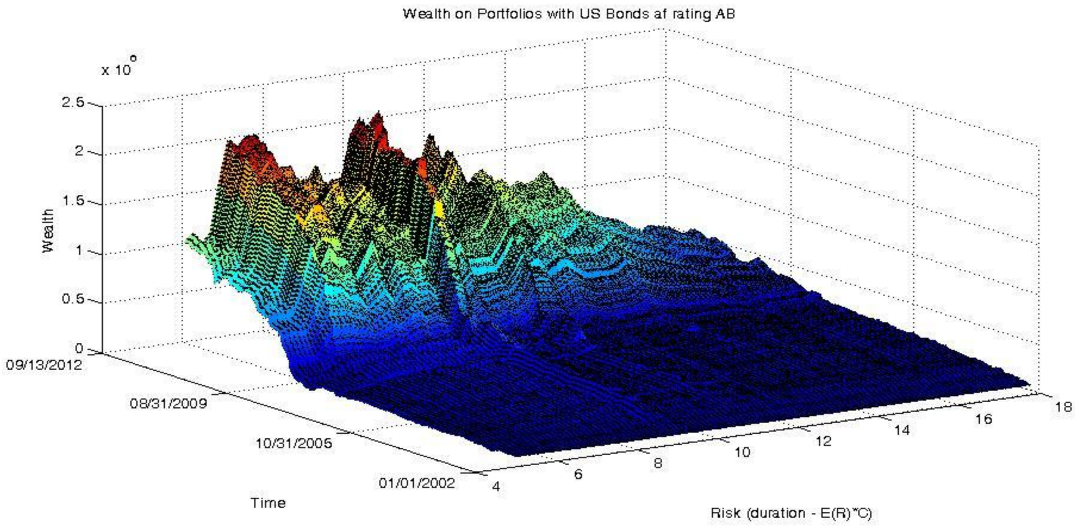
Figure 1 Ex-post wealth for bonds in the credit risk classes of type AB - Redington



Source: author's calculations using data from DataSteam

In the Figure 2 we observe the results of the portfolio immunization in average problem. Figure 2 illustrates that we obtain the best results for medium-term duration portfolio since we have a wealth greater than 1,500,000 USD for portfolio with a risk between 7 and 11. These results are generally higher than what we obtain with the classical strategy in which we have the worst performance for medium-term duration. Moreover it is evident that portfolios with longest or shortest risk values show smaller levels of wealth. This leads to the conclusion that the new proposed approach generally performs better for portfolios with medium durations but it is not so suitable for short or long risk levels portfolio strategies. However, in this case we consider bonds with a very large classes of ratings but these results are absolutely relevant since we can manage different risk measures for different investment periods.

Figure 2 Ex-post wealth for bonds in the credit risk classes of type AB – Immunization Average



Source: author's calculations using data from DataSteam

3.2.2 Portfolio selection that account the market risk

In order to reduce the market risk we suggest to maximize two performance measures (Sharpe ratio and Rachev ratio) on the 60 funds we computed in the portfolio problems of Section 3.2.1. In particular, we adopt the parameters $\alpha = \beta = 0.03$ for the Rachev ratio.

Moreover, we recalibrate the portfolio weekly and we use a moving window of 6 months of historical observations (125 trading days) to compute the performance measures. Thus we recalibrate the portfolio 532 times starting at date 30 June 2002. Doing so, at the *k-th* optimization ($k = 1, 2, \ldots, 532$) two main steps are performed to compute the ex-post final wealth:

Step 1 - Determine the optimal portfolio x that maximizes the performance measure, i.e. the solution of the following optimization problem:

$$max_x \, G(x'r)$$
$$subject\ to \quad \begin{cases} \sum_{i=1}^{n} x_i = 1 \\ x_i \geq 0 \\ x_i \leq 0.21 \end{cases} \tag{19}$$

To face the portfolio problem we should account of the computationally complexity. We observe that:

a) the maximization of the Sharpe Ratio can be solved as a quadratic-type problem and thus it presents a unique solution;
b) the optimization of Rachev Ratio does not give an unique global optimum, thus to overcome the computational complexity problem for global maximum, we use the heuristic proposed by Angelelli and Ortobelli (2009) that presents significant improvements in terms of objective function and portfolio weights with respect to the classic function fmincon provided with the optimization toolbox of MATLAB.

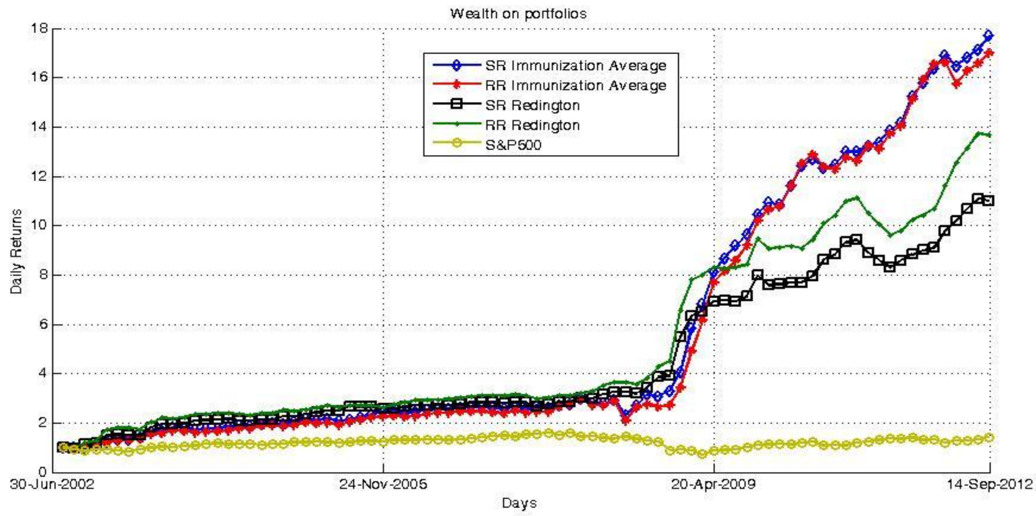
Step 2 - Compute the ex-post final wealth (not subtracting anymore the transaction costs).

Steps 1 and 2 are repeated for both the performance measures (Sharpe ratio and the Rachev ratio) some observations are available.

In this case we don't apply liquidity constraints and transaction costs since they were applied in the previous portfolio selection analysis. The result is reported in Figures 3. In particular, we compare the performance of the stock index S&P500 (red line) with the wealth obtained maximizing either the Rachev Ratio (blue line) or the Sharpe Ratio (green line).

In all cases we notice that the optimal portfolios are well diversified and the average of the number of funds used is always greater than 10 (from the 60 funds obtained for each credit risk class). Moreover, we also observe that this portfolio selection reduces significantly the market risk and the crisis impact on the portfolio wealth is very low.

Figure 3 Ex-post wealth for bonds in the credit risk classes of type AB



Source: author's calculations using data from DataSteam

The optimization of the Rachev Ratio in the classical approach produces an higher wealth: 1,400,000 USD. The comparison with S&P500 is clear. If we invest in equity and we start with an initial wealth of 100,000 USD we obtain a final wealth of 150,000 USD (with a peak of 161,000 USD). The results obtained with the new approach, however, show an higher performance than what we obtain with the classical Redington method. This fact is confirmed by the final value in both strategies: using Sharpe ratio or Rachev ratio. In these cases the final wealth is more than 1,600,000 USD and the paths of the two ratios of performance are fairly similar during the overall period.

Thus the common belief that in the market there is a risk premium is countered by the evidence that the bond portfolio returns are often greater than equity returns with less volatility. In non crisis periods equity provides a better result than high rating bonds portfolio while the behavior is similar to the low rating bonds portfolio. But in crisis periods the best asset class choice is glaring: bonds.

During the 2002 and 2008-2009 crisis the stock index collapse and the capital flows move to the more safety fixed income market. This phenomenon is interpretable as a speculative bubble on the bonds market.

4. Conclusion

In this paper we discuss and analyze the portfolio selection problem in the fixed income market. The focus of the paper is the reduction of price risk and market risk that are two of the main risk sources that influence the bond behavior. Therefore we suggest to determine the optimal portfolios in separated steps where we clean and control each risk source. In particular we use a two stages model applied on a dynamic dataset clustered by rating classes and liquidity levels. In the first phase we control the price risk of bonds considering 60 optimal funds with a fixed risk measure. Moreover, we introduce an immunization in average approach in addition to the classical Redington immunization problem to lead a new risk measure in the portfolio optimization problem. The results show that the final wealth is maxima for different portfolio durations between the two risk measure.

In the second phase we control the market risk optimizing a performance measure (Sharpe Ratio and Rachev Ratio) on the 60 funds.

The ex-post results show that bonds portfolios perform very well even during the 2007-2012 market crisis. The ex-post wealth obtained in the immunization in average approach is higher than the other case without distinction in the selected ratio of performance. Furthermore this empirical comparison show that it is possible to have a low risk portfolio and jointly very high returns during periods of market crisis. In particular the comparison with the equity index shows that during the crisis there is a speculative bubble on the fixed income market because the capital flows goes to more safety assets, the so called fly to safety. This effect is more strong for low rating bonds seen as the best equity substitute but anyway less risky.

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